

Shear-free Anisotropic Cosmological Models in $f(R)$ Gravity

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Abstract We study a class of shear-free, homogeneous but anisotropic cosmological models with imperfect matter sources in the context of $f(R)$ gravity. We show that the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other. We also show that within the class of orthogonal $f(R)$ models, small perturbations of shear are damped, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion as well as the heat flux of the curvature fluid. Specializing in locally rotationally symmetric spacetimes in orthonormal frames, we examine the late-time behaviour of the de Sitter universe in $f(R)$ gravity. For the Starobinsky model of $f(R)$, we study the evolutionary behaviour of the Universe by numerically integrating the Friedmann equation, where the initial conditions for the expansion, acceleration and jerk parameters are taken from observational data.

Keywords shear-free spacetimes · homogeneity · cosmic anisotropy · modified gravity · expansion · inflation · cosmological perturbations

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1 Introduction

As a result of the current understanding that the Universe is in a state of accelerated expansion, many modifications to General Relativity (GR), the theory on which modern cosmology is based, have been proposed recently. One such modification consists of a class of higher-order gravity models that attempt to address the shortcomings of GR in the infrared (IR) and ultraviolet (UV) ranges [1, 2, 3, 4, 5, 6, 7, 8]. These models are generally obtained by including higher-order curvature invariants in the Einstein-Hilbert action, by making the action nonlinear in the Ricci curvature R , or contain terms involving combinations of derivatives of R , in which case the models are known as $f(R)$ theories of gravity.

First proposed by Buchdal [9], $f(R)$ theories gained more popularity after further developments by Starobinsky [10] and later following the realization of the discrepancy between theory and observation [11, 12, 13, 5, 14, 15].

The role of shear in general relativistic [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] and $f(R)$ cosmologies [27, 28, 29, 30, 31, 32, 33, 34] has been the subject of intense study for some time now, with the studies focusing mostly on the special nature of shear-free cases. In particular, it was shown in [16] that in the orthogonally spatially homogeneous models with vanishing shear, the anisotropic stresses are related to the anisotropic curvature of the spatial hypersurface through the electric part of the Weyl tensor. It was also shown that within the class of orthogonal models, small perturbations of shear are damped, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion.

The main focus of this work is the analysis of anisotropic but homogeneous, shear-free models whose underlying theory of gravitational interaction is $f(R)$ -gravity.

The rest of this paper is organised as follows: in Sec. 2 a covariant description of $f(R)$ field equations is presented. In Sec. 3 we specialise to orthogonal cosmological models with anisotropic matter sources and analyse the properties of such models in the case of shear-free imperfect fluids in Sec. 4. In Sec. 5, the analysis is taken further by considering subclasses of locally rotationally symmetric spacetimes with barotropic equations of state and a qualitative analysis of such models has been made. Finally in Sec. 6 we discuss the results and give conclusions.

Natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, and Latin indices run from 0 to 3. The symbols ∇ , $\tilde{\nabla}$ and the overdot \cdot represent the usual covariant derivative, the spatial covariant derivative, and differentiation with respect to cosmic time. We use the $(-+++)$ spacetime signature and the Riemann tensor is defined by

$$R_{bcd}^a = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{bd}^e \Gamma_{ce}^a - \Gamma_{bc}^f \Gamma_{df}^a ,$$

where the Γ_{bd}^a are the Christoffel symbols (i.e., symmetric in the lower indices), defined by

$$\Gamma_{bd}^a = \frac{1}{2} g^{ae} (g_{be,d} + g_{ed,b} - g_{bd,e}) .$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor:

$$R_{ab} = g^{cd} R_{cadb} .$$

The completely anti-symmetric pseudotensor η^{abcd} is defined such that

$$\eta_{0123} = \sqrt{-g} ,$$

where $g = \det(g_{ab})$ is the determinant of the metric g_{ab} .

Unless otherwise stated, primes ' etc are shorthands for derivatives with respect to the Ricci scalar

$$R = R^a_a$$

and f is used as a shorthand for $f(R)$. Moreover the following standard notations are used:

- (ab) : symmetrization over the indices a and b ,
- $[ab]$: anti-symmetrization over the indices a and b ,
- $\langle ab \rangle$: orthogonal, symmetric, trace-free projection over the indices a and b .

2 Covariant Description of the Field Equations

In the standard $f(R)$ -gravity formulation, one starts with the modified Einstein-Hilbert action

$$\mathcal{A} = \frac{1}{2} \int d^4x \sqrt{-g} [f(R) + 2\mathcal{L}_m] , \quad (1)$$

where \mathcal{L}_m stands for the matter field contribution to the Lagrangian, and uses the variational principle of least action with respect to the metric g_{ab} to obtain the generalised Einstein Field Equations (EFEs)

$$G_{ab} = \tilde{T}_{ab}^m + T_{ab}^R \equiv T_{ab} . \quad (2)$$

Here we have defined

$$\tilde{T}_{ab}^m \equiv \frac{T_{ab}^m}{f'} , \quad T_{ab}^R \equiv \frac{1}{f'} \left[\frac{1}{2} (f - Rf') g_{ab} + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right] \quad (3)$$

as the effective matter and curvature energy-momentum tensors (EMTs), respectively. The EMT of standard matter is given by

$$T_{ab}^m = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{ab}} = \mu_m u_a u_b + p_m h_{ab} + q_a^m u_b + q_b^m u_a + \pi_{ab}^m , \quad (4)$$

where μ_m , p_m , q_a^m and π_{ab}^m are the associated energy density, isotropic pressure, heat flux and anisotropic pressure, respectively, and $u^a \equiv \frac{dx^a}{dt}$ is the normalized 4-velocity of fundamental observers comoving with the fluid. We use this vector to define the covariant time derivative for any tensor $S_{c..d}^{a..b}$ along an observer's worldlines:

$$\dot{S}_{c..d}^{a..b} = u^e \nabla_e S_{c..d}^{a..b}. \quad (5)$$

On the other hand, we use the projection tensor $h_{ab} \equiv g_{ab} + u_a u_b$ to define the fully orthogonally projected covariant derivative for any tensor $S_{c..d}^{a..b}$:

$$\tilde{\nabla}_e S_{c..d}^{a..b} = h_f^a h_c^p \dots h_g^b h_d^q h_e^r \nabla_r S_{p..q}^{f..g}, \quad (6)$$

with total projection on all the free indices. We extract the orthogonally projected symmetric trace-free part of vectors and rank-2 tensors using

$$V^{(a)} = h_b^a V^b, \quad S^{(ab)} = \left[h_c^{(a} h_d^{b)} - \frac{1}{3} h^{ab} h_{cd} \right] S^{cd}, \quad (7)$$

and the volume element for the restspaces orthogonal to u^a is given by [35]

$$\varepsilon_{abc} = u^d \eta_{dabc} = -\sqrt{|g|} \delta_a^0 \delta_b^1 \delta_c^2 \delta_d^3 u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]}, \quad \varepsilon_{abc} u^c = 0, \quad (8)$$

where η_{abcd} is the 4-dimensional volume element satisfying the conditions

$$\eta_{abcd} = \eta_{[abcd]} = 2\varepsilon_{ab[c} u_{d]} - 2u_{[a} \varepsilon_{b]cd}. \quad (9)$$

The covariant spatial divergence and curl of vectors and rank-2 tensors are given as [36]

$$\text{div} V = \tilde{\nabla}^a V_a, \quad (\text{div} S)_a = \tilde{\nabla}^b S_{ab}, \quad (10)$$

$$\text{curl} V_a = \varepsilon_{abc} \tilde{\nabla}^b V^c, \quad \text{curl} S_{ab} = \varepsilon_{cd(a} \tilde{\nabla}^c S_{b)}^d. \quad (11)$$

The 4-velocity vector field u^a can be split into its irreducible parts as follows

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c, \quad (12)$$

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{(a} u_{b)}$ and $\omega^a \equiv \varepsilon^{abc} \tilde{\nabla}_b u_c$.

We can also split the Weyl conformal curvature tensor [35, 37]

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a} R^{b]}_{[c} g^{d]} + \frac{R}{3} g^{[a} g^{b]}_{[c} g^{d]} \quad (13)$$

into its “gravito-electric” (GE) and “gravito-magnetic” (GM) parts, respectively, as

$$E_{ab} \equiv C_{agbh} u^g u^h, \quad H_{ab} \equiv \frac{1}{2} \eta_{ae}{}^{gh} C_{ghbd} u^e u^d. \quad (14)$$

The GE and GM components represent the free gravitational field [35] and they describe gravitational action at a distance - tidal forces and gravitational waves. They influence the motion of matter and radiation through the geodesic deviation for timelike and null-vector fields, respectively.

The total energy density, isotropic and anisotropic pressures and heat flux of the $f(R)$ universe are given, respectively, by [38]

$$\mu \equiv \frac{\mu_m}{f'} + \mu_R, \quad p \equiv \frac{p_m}{f'} + p_R, \quad \pi_{ab} \equiv \frac{\pi_{ab}^m}{f'} + \pi_{ab}^R, \quad q_a \equiv \frac{q_a^m}{f'} + q_a^R, \quad (15)$$

where the thermodynamic quantities for the *curvature fluid* component are defined as

$$\mu_R = \frac{1}{f'} \left[\frac{1}{2} (Rf' - f) - \Theta f'' \dot{R} + f'' \tilde{\nabla}^2 R \right], \quad (16)$$

$$p_R = \frac{1}{f'} \left[\frac{1}{2} (f - Rf') + f'' \ddot{R} + f''' \dot{R}^2 + \frac{2}{3} \left(\Theta f'' \dot{R} - f'' \tilde{\nabla}^2 R - f''' \tilde{\nabla}^a R \tilde{\nabla}_a R \right) \right], \quad (17)$$

$$q_a^R = -\frac{1}{f'} \left[f''' \dot{R} \tilde{\nabla}_a R + f'' \tilde{\nabla}_a \dot{R} - \frac{1}{3} f'' \Theta \tilde{\nabla}_a R \right], \quad (18)$$

$$\pi_{ab}^R = \frac{1}{f'} \left[f'' \tilde{\nabla}_{(a} \tilde{\nabla}_{b)} R + f''' \tilde{\nabla}_{(a} R \tilde{\nabla}_{b)} R - \sigma_{ab} \dot{R} f'' \right]. \quad (19)$$

In the 1 + 3 covariant decomposition [39, 40], a fundamental observer slices spacetime into time and space. The Bianchi and Ricci identities

$$\nabla_{[a} R_{bc]d}{}^e = 0, \quad (\nabla_a \nabla_b - \nabla_b \nabla_a) u_c = R_{abc}{}^d u_d \quad (20)$$

applied on the total fluid 4-velocity u^a result in evolution equations-which propagate consistent initial data on some initial ($t = t_0$) hypersurface S_0 uniquely along timelike congruences- and constraint equations- which restrict the initial data to be specified [41]. In $f(R)$ gravity, the evolution equations

are given by [38]

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \tilde{\nabla}^a q_a^m - 2A_a q_a^m - \sigma_b^a \pi_{a,m}^b, \quad (21)$$

$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f'^2} \dot{R} - \tilde{\nabla}^a q_a^R - 2A_a q_a^R - \sigma_b^a \pi_{a,R}^b, \quad (22)$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) + \tilde{\nabla}_a A^a - A_a A^a - \sigma_{ab} \sigma^{ab} + 2\omega_a \omega^a, \quad (23)$$

$$\dot{q}_a^m = -\frac{4}{3}\Theta q_a^m - (\mu_m + p_m)A_a - \tilde{\nabla}_a p_m - \tilde{\nabla}^b \pi_{ab}^m - \sigma_a^b q_b^m - A^b \pi_{ab}^m - \varepsilon_{abc} \omega^b q_a^c, \quad (24)$$

$$\begin{aligned} \dot{q}_a^R = & -\frac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f'^2} \tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi_{ab}^R - \sigma_a^b q_b^R \\ & - (\mu_R + p_R)A_a - A^b \pi_{ab}^R - \varepsilon_{abc} \omega^b q_a^c, \end{aligned} \quad (25)$$

$$\dot{\omega}_a = -\frac{2}{3}\Theta \omega_a - \frac{1}{2}\varepsilon_{abc} \tilde{\nabla}^b A^c + \sigma_a^b \omega_b, \quad (26)$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta \sigma_{ab} - E_{ab} + \frac{1}{2}\pi_{ab} + \tilde{\nabla}_{\langle a} A_{b\rangle} + A_{\langle a} A_{b\rangle} - \sigma_{\langle a}^c \sigma_{b\rangle c} - \omega_{\langle a} \omega_{b\rangle}, \quad (27)$$

$$\begin{aligned} \dot{E}_{ab} + \frac{1}{2}\dot{\pi}_{ab} = & \varepsilon_{cd\langle a} \tilde{\nabla}^c H_{b\rangle}^d - \Theta (E_{ab} + \frac{1}{6}\pi_{ab}) - \frac{1}{2}(\mu + p)\sigma_{ab} - \frac{1}{2}\tilde{\nabla}_{\langle a} q_{b\rangle} \\ & + 3\sigma_{\langle a}^c (E_{b\rangle c} - \frac{1}{6}\pi_{b\rangle c}) - A_{\langle a} q_{b\rangle} + \varepsilon_{cd\langle a} \left[2A^c H_{b\rangle}^d + \omega^c (E_{b\rangle}^d + \frac{1}{2}\pi_{b\rangle}^d) \right], \end{aligned} \quad (28)$$

$$\begin{aligned} \dot{H}_{ab} = & -\Theta H_{ab} - \varepsilon_{cd\langle a} \tilde{\nabla}^c E_{b\rangle}^d + \frac{1}{2}\varepsilon_{cd\langle a} \tilde{\nabla}^c \pi_{b\rangle}^d \\ & + 3\sigma_{\langle a}^c H_{b\rangle c} + \frac{3}{2}\omega_{\langle a} q_{b\rangle} - \varepsilon_{cd\langle a} \left[2A^c E_{b\rangle}^d - \frac{1}{2}\sigma_{b\rangle}^c q^d - \omega^c H_{b\rangle}^d \right], \end{aligned} \quad (29)$$

whereas the constraints read

$$(C^1)_a := \tilde{\nabla}^b \sigma_{ab} - \frac{2}{3}\tilde{\nabla}_a \Theta + \varepsilon_{abc} \left(\tilde{\nabla}^b \omega^c + 2A^b \omega^c \right) + q_a = 0, \quad (30)$$

$$(C^2)_{ab} := \varepsilon_{cd\langle a} \tilde{\nabla}^c \sigma_{b\rangle}^d + \tilde{\nabla}_{\langle a} \omega_{b\rangle} - H_{ab} - 2A_{\langle a} \omega_{b\rangle} = 0, \quad (31)$$

$$\begin{aligned} (C^3)_a := & \tilde{\nabla}^b H_{ab} + (\mu + p)\omega_a + \varepsilon_{abc} \left[\frac{1}{2}\tilde{\nabla}^b q^c + \sigma_{bd} (E^d_c + \frac{1}{2}\pi^d_c) \right] \\ & + 3\omega_b (E^{ab} - \frac{1}{6}\pi^{ab}) = 0, \end{aligned} \quad (32)$$

$$\begin{aligned} (C^4)_a := & \tilde{\nabla}^b E_{ab} + \frac{1}{2}\tilde{\nabla}^b \pi_{ab} - \frac{1}{3}\tilde{\nabla}_a \mu + \frac{1}{3}\Theta q_a \\ & - \frac{1}{2}\sigma_a^b q_b - 3\omega^b H_{ab} - \varepsilon_{abc} [\sigma^{bd} H_d^c - \frac{3}{2}\omega^b q^c] = 0, \end{aligned} \quad (33)$$

$$(C^5) := \tilde{\nabla}^a \omega_a - A_a \omega^a = 0. \quad (34)$$

The Gauß-Codazzi equations are given by

$$\tilde{R}_{ab} + \dot{\sigma}_{\langle ab\rangle} + \Theta \sigma_{ab} - \tilde{\nabla}_{\langle a} A_{b\rangle} - A_{\langle a} A_{b\rangle} - \pi_{ab} - \frac{1}{3} \left(2\mu - \frac{2}{3}\Theta^2 \right) h_{ab} = 0, \quad (35)$$

where \tilde{R}_{ab} is the Ricci tensor on 3-D spatial hypersurfaces with $\tilde{R} = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2$ as its corresponding (3-curvature) Ricci scalar.

3 Orthogonal Models

Following [16], the orthogonal models are characterised by the matter EMT representing an anisotropic fluid without heat fluxes

$$T_{ab}^m = \mu_m u_a u_b + p_m h_{ab} + \pi_{ab}^m, \quad (36)$$

the matter energy density and isotropic pressure measured by an observer moving with the velocity u^a . In this setting, we have an irrotational and non-accelerated flow of the vector field u^a and therefore $\omega_a = 0 = A_a$. Thus the corresponding evolution and constraint equations are given by

$$\dot{\mu}_m = -(\mu_m + p_m)\Theta - \sigma_b^a \pi_{a,m}^b, \quad (37)$$

$$\dot{\mu}_R = -(\mu_R + p_R)\Theta + \frac{\mu_m f''}{f r^2} \dot{R} - \tilde{\nabla}^a q_a^R - \sigma_b^a \pi_{a,R}^b, \quad (38)$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p) - \sigma_{ab}\sigma^{ab}, \quad (39)$$

$$\dot{q}_a^R = -\frac{4}{3}\Theta q_a^R + \frac{\mu_m f''}{f r^2} \tilde{\nabla}_a R - \tilde{\nabla}_a p_R - \tilde{\nabla}^b \pi_{ab}^R - \sigma_a^b q_b^R, \quad (40)$$

$$\dot{\sigma}_{ab} = -\frac{2}{3}\Theta \sigma_{ab} - E_{ab} + \frac{1}{2}\pi_{ab} - \sigma_{\langle a}^c \sigma_{b\rangle c}, \quad (41)$$

$$\begin{aligned} \dot{E}_{ab} + \frac{1}{2}\dot{\pi}_{ab} &= \varepsilon_{cd\langle a} \tilde{\nabla}^c H_{b\rangle}^d - \Theta \left(E_{ab} + \frac{1}{6}\pi_{ab} \right) - \frac{1}{2}(\mu + p)\sigma_{ab} - \frac{1}{2}\tilde{\nabla}_{\langle a} q_{b\rangle}^R \\ &\quad + 3\sigma_a^{\langle c} (E_{b\rangle c} - \frac{1}{6}\pi_{b\rangle c}), \end{aligned} \quad (42)$$

$$\begin{aligned} \dot{H}_{ab} &= -\Theta H_{ab} - \varepsilon_{cd\langle a} \tilde{\nabla}^c E_{b\rangle}^d + \frac{1}{2}\varepsilon_{cd\langle a} \tilde{\nabla}^c \pi_{b\rangle}^d + 3\sigma_a^{\langle c} H_{b\rangle c} \\ &\quad + \frac{1}{2}\varepsilon_{cd\langle a} \sigma_{b\rangle}^c q_R^d, \end{aligned} \quad (43)$$

$$(C^{*1})_a := \tilde{\nabla}^b \sigma_{ab} - \frac{2}{3}\tilde{\nabla}_a \Theta + q_a^R = 0, \quad (44)$$

$$(C^{*2})_{ab} := \varepsilon_{cd\langle a} \tilde{\nabla}^c \sigma_{b\rangle}^d - H_{ab} = 0, \quad (45)$$

$$(C^{*3})_a := \tilde{\nabla}^b H_{ab} + \varepsilon_{abc} \left[\frac{1}{2}\tilde{\nabla}^b q_R^c + \sigma_{bd} \left(E^d{}_c + \frac{1}{2}\pi^d{}_c \right) \right] = 0, \quad (46)$$

$$\begin{aligned} (C^{*4})_a &:= \tilde{\nabla}^b E_{ab} + \frac{1}{2}\tilde{\nabla}^b \pi_{ab} - \frac{1}{3}\tilde{\nabla}_a \mu + \frac{1}{3}\Theta q_a^R - \frac{1}{2}\sigma_a^b q_b \\ &\quad - \varepsilon_{abc} \sigma^{bd} H_d^c = 0, \end{aligned} \quad (47)$$

where the new equations corresponding to Eqns (26) and (34) become trivial, and with Eqn (24) resulting in the constraint

$$(C^{*5})_a := \tilde{\nabla}_a p_m + \tilde{\nabla}^b \pi_{ab}^m = 0. \quad (48)$$

4 Shear-free Anisotropic Models with an Imperfect Fluid

From causal relativistic thermodynamical relationships for imperfect fluids, the anisotropic pressure is known to evolve according to [42, 43, 44, 45]

$$\tau \dot{\pi}_{ab} + \pi_{ab} = -\lambda \sigma_{ab}, \quad (49)$$

where τ and λ are relaxation and viscosity parameters. If we consider cases where τ is negligible and λ is a positive constant, and use the fairly popular ansatz (valid near thermal equilibrium, such as in the very early stages of the Universe) for the equation of state [25, 16, 46]

$$\pi_{ab} = -\lambda \sigma_{ab}, \quad (50)$$

then Eqns (15) and (19) imply that we can rewrite (50) as

$$\pi_{ab}^m + f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R + f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b \rangle} R = \sigma_{ab} \left(\dot{R} f'' - \lambda f' \right) . \quad (51)$$

For a general case of vanishing shear tensor during the entire cosmic evolution, one can see from Eqn (51) that

$$\pi_{ab}^m = -f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R - f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b \rangle} R . \quad (52)$$

Moreover, the Gauß-Codazzi equations (35) reduce to

$$\tilde{R}_{ab} - \frac{1}{3} \tilde{R} h_{ab} = \pi_{ab} = \frac{1}{f'} \left(\pi_{ab}^m + f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R + f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b \rangle} R \right) , \quad (53)$$

thus showing that even if the matter anisotropic stress vanishes, no constant-curvature geometries are guaranteed and hence no necessarily FLRW universes. It is also worth noticing that, unlike in GR, if we allow the matter anisotropic pressure to be nonzero despite a vanishing shear, constant-curvature models are allowed provided

$$f'' \tilde{\nabla}_{\langle a} \tilde{\nabla}_{b \rangle} R + f''' \tilde{\nabla}_{\langle a} R \tilde{\nabla}_{b \rangle} R = 0 . \quad (54)$$

The converse also holds, i.e., it is possible, unlike in GR, to have a vanishing matter anisotropic pressure π_{ab}^m for a non-constant curvature geometry.

One can see the tidal effect on the anisotropic stresses by dropping the shear terms of Eqn (41), obtaining the equation

$$\pi_{ab} = 2E_{ab} , \quad (55)$$

which shows that, in this case as in GR [16], the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other, a necessary and sufficient condition for the shear to remain zero if initially vanishing.

If the shear is nonzero, but with very small second-order contributions, then one can show that Eqn (41) can be approximated by

$$\dot{\sigma}_{ab} \approx -\frac{2}{3} \Theta \sigma_{ab} . \quad (56)$$

Rewriting Eqn (56) as

$$(\sigma^2)^\cdot \approx -\frac{4}{3} \Theta \sigma^2 \quad (57)$$

shows that the shear decays with expansion. One can, therefore, conclude that *within the class of orthogonal $f(R)$ models, small perturbations of shear are damped*, i.e. that these models are stable if expanding, a result similar to that obtained in [16] for models whose underlying theory is GR.

For shear-free orthogonal models satisfying Eqn (55), we see that Eqn (45) implies a purely electric Weyl tensor, i.e., $H_{ab} = 0$, and hence Eqn (43) reduces to an identity:

$$\varepsilon_{cd\langle a} \tilde{\nabla}^c E_{b \rangle}^d = \frac{1}{2} \varepsilon_{cd\langle a} \tilde{\nabla}^c \pi_{b \rangle}^d . \quad (58)$$

Moreover, it is straightforward to show using Eqns (42) and (47) that

$$\dot{E}_{ab} = -\frac{2}{3}\Theta E_{ab} - \frac{1}{4}\tilde{\nabla}_{\langle a}q_{b\rangle}^R, \quad (59)$$

$$\tilde{\nabla}^b E_{ab} = \frac{1}{6}\left(\tilde{\nabla}_a \mu - \frac{1}{3}\Theta q_a^R\right). \quad (60)$$

Defining $E^2 \equiv E_{ab}E^{ab}$, we can rewrite Eqn (59) as

$$(E^2)^\cdot = -\frac{4}{3}\Theta E^2 - \frac{1}{8}\left(\tilde{\nabla}_{\langle a}q_{b\rangle}^R E^{ab} + \tilde{\nabla}^{\langle a}q_{R}^{\rangle b} E_{ab}\right), \quad (61)$$

thus showing the decay of the electric part of the Weyl tensor and the anisotropic stress tensor with the expansion. This equation also implies decay with the heat flux of the curvature fluid if the bracketed terms in the r.h.s are overall positive.

Let us now consider the generalized Friedman equation

$$\Theta^2 = 3\left(\mu - \frac{1}{2}\tilde{R}\right). \quad (62)$$

Since the total energy density μ is not always guaranteed to be positive for generic $f(R)$ models, it is not straightforward to comment on the asymptotic isotropization of expanding shear-free anisotropic models for the different values of the spatial curvature. This is in contrast to the GR result where, for example, expanding shear-free models which exhibit negative spatial curvature asymptotically approach isotropy [16].

5 Anisotropic LRS Models

Let us consider the locally rotationally symmetric (LRS) metric given by

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)[d\theta^2 + f^2(\theta)d\phi^2], \quad (63)$$

where

$$f(\theta) = \begin{cases} \sin(\theta) & \text{for } \tilde{R} > 0 \text{ (Kantowski-Sachs),} \\ \theta & \text{for } \tilde{R} = 0 \text{ (Bianchi I),} \\ \sinh(\theta) & \text{for } \tilde{R} < 0 \text{ (Bianchi III).} \end{cases}$$

Here $\tilde{R} = 2k/b^2$ for $k = \pm 1, 0$. The non-vanishing kinematic quantities for these models are the expansion and shear, respectively given as

$$\Theta = \frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}, \quad (64)$$

$$2\sigma^2 = \sigma_{ab}\sigma^{ab} = \frac{1}{\sqrt{3}}\left(\frac{\dot{a}}{a} - \frac{\dot{b}}{b}\right). \quad (65)$$

Consider the EMT of the imperfect fluid matter source to be of the form

$$T_{ab} = \mu u_a u_b + \bar{p} h_{ab} - \bar{\pi}(e_1)_a (e_1)_b \quad (66)$$

where, because of the rotational symmetry, $e_1 = \frac{1}{a} \frac{\partial}{\partial r}$ is defined as the unit vector along the axis of symmetry. Whereas μ represents the total energy density measured by a comoving observer, the pressure measured by the same observer is

$$p = \bar{p} - \bar{\pi} . \quad (67)$$

Here the anisotropic stress tensor in the orthonormal tetrad bases

$$e_0 = \frac{\partial}{\partial t} , \quad e_1 = \frac{1}{a} \frac{\partial}{\partial r} , \quad e_2 = \frac{1}{b} \frac{\partial}{\partial \theta} , \quad e_3 = \frac{1}{b \sin \theta} \frac{\partial}{\partial \phi} \quad (68)$$

is given by

$$\pi_{ab} = \text{diag} \left(0, -\frac{2}{3} \bar{\pi}, \frac{1}{3} \bar{\pi}, \frac{1}{3} \bar{\pi} \right) . \quad (69)$$

This way we can write the modified EFEs as

$$2 \frac{\dot{a}\dot{b}}{ab} + \frac{k + \dot{b}^2}{b^2} = \mu , \quad (70)$$

$$2 \frac{\ddot{b}}{b} + \frac{k + \dot{b}^2}{b^2} = -\bar{p} + \bar{\pi} , \quad (71)$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} = -\bar{p} , \quad (72)$$

whereas the conservation equations (37), (38), (40) and (48) are rewritten as

$$\dot{\mu}_m = - \left(\mu_m + \bar{p}_m - \frac{1}{3} \bar{\pi}_m \right) \Theta , \quad (73)$$

$$\dot{\mu}_R = - \left(\mu_R + \bar{p}_R - \frac{1}{3} \bar{\pi}_R \right) \Theta + \frac{\mu_m f''}{f r^2} \dot{R} - \tilde{\nabla}^a q_a^R , \quad (74)$$

$$\dot{q}_a^R = -\frac{4}{3} \Theta q_a^R + \frac{\mu_m f''}{f r^2} \tilde{\nabla}_a R - \tilde{\nabla}_a \bar{p}_R - \tilde{\nabla}_a \bar{\pi}_R , \quad (75)$$

$$\tilde{\nabla}_a \bar{p}_m (e_1)^a = \tilde{\nabla}_a \bar{\pi}_m (e_1)^a . \quad (76)$$

As a result of the homogeneity assumption, $\bar{\pi} = \bar{\pi}_m(t)$ and therefore Eqn(76) is trivially satisfied.

We notice from Eqn (65) that for the case of vanishing shear, $a(t) = b(t)$ and thus the modified EFES (70)-(72) reduce to

$$3 \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu , \quad (77)$$

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -\bar{p} + \bar{\pi} , \quad (78)$$

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\bar{p} . \quad (79)$$

Subtracting Eqn (79) from Eqn (78) yields

$$\bar{\pi} = \frac{k}{a^2} , \quad (80)$$

and therefore

$$E_{ab} = \text{diag}(0, -2E, E, E) , \quad (81)$$

where $E = \frac{\bar{\pi}}{6}$.

We adopt the barotropic EoS , $p_m = (\gamma_m - 1)\mu_m$, where $p_m = \bar{p}_m - \bar{\pi}_m/3$, from the continuity Eqn (73) for p_m , we obtain $\mu_m = \mu_m^0 a^{-3\gamma_m}$. To integrate (77) we need to know μ_R . Indeed, it is a hard job to integrate (74) although we are working in the homogeneous case. But we can rewrite (77) in the following form:

$$3\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \mu_m^0 a^{-3\gamma_m} + \frac{1}{f'} \left[\frac{1}{2}(Rf' - f) - \frac{3\dot{a}}{a} f'' \dot{R} \right] . \quad (82)$$

Here μ_m^0 is the matter density at the time $t = t_0$ and γ_m is the EoS parameter for the matter content. As we see, Eqn (82) is model dependent. To specify solutions we must choose a specific model of $f(R)$ gravity. Otherwise, we cannot integrate it explicitly. Let us have a brief qualitative analysis of (82). If we are looking for the late-time behavior of the solutions for (82) and if we suppose that the space is flat $k = 0$, and without matter, the evolution is defined by the de Sitter (dS) solution, in which we put $R = 6H_0^2$, where H_0 is the time scale of the dS universe. In this simple case, we can solve Eqn (82) to obtain:

$$H_0^2 = \frac{1}{6f'}(Rf' - f). \quad (83)$$

But this is not the only case we can solve (77). Suppose that we choose a model of $f(R)$, so (77) reduces generally to a fourth-order ODE, which can be solved in terms of quadratures. For example, in the so-called Starobinsky model, $f(R) = R + \alpha R^2$, which is motivated for the inflationary universe scenario [10], Eqn (82) reduces to the following differential equation:

$$3\frac{\dot{a}^2}{a^2} = \mu_m^0 a^{-3\gamma_m} + \frac{\alpha}{2} \frac{R^2 - 12H\dot{R}}{1 + 2\alpha R} , \quad (84)$$

where $R = 6 \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right)$. Eqn (84) is a third order ODE for $a(t)$. So we need to specify initial condition(s) (ICs), as well as integrability condition(s). The cosmological ICs are fitted using the Hubble H , deceleration q , jerk j , and snap s parameters evaluated at the present time $t = t_0$. We can adjust the first derivatives of the scale factor as $a(0) = a_0 = 1$, $\dot{a}(0) = H_0 a_0$, $\ddot{a}(0) = -H_0^2 a_0 q_0$, $\ddot{\ddot{a}}(0) = H_0^3 j_0 a_0^{-1}$ where q_0 is the deceleration parameter at the initial time (present time), j_0 is the jerk parameter at the instant $t = t_0$, etc [47]. Fortunately, these data have been measured with high precisions.

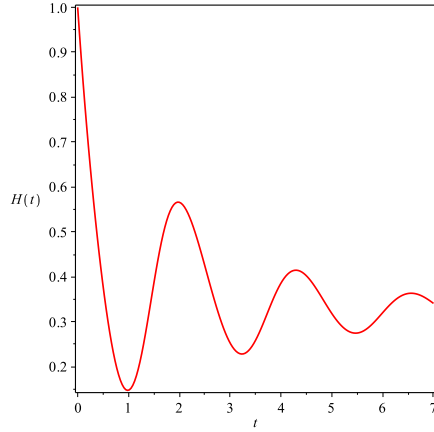


Fig. 1 Numerical solution for $H(t)$. The model of $f(R)$ is the one proposed by Starobinsky, with $\alpha = 0.02$. The cosmological data are fitted with observational data for extended cosmological parameters.

A series solution for $a(t)$ in Eqn (84) has been developed using these cosmographic parameters which are all evaluated at $t = t_0$:

$$\begin{aligned}
 a(t) = & 1 + H_0(t - t_0) - \frac{1}{2}H_0^2 q_0(t - t_0)^2 \\
 & - \frac{1}{216} \frac{(-3H_0^2 + 54H_0^4\alpha + \mu_m + 12\alpha\mu_m H_0^2 - 12\alpha\mu_m^0 H_0^2 q_0 + 18\alpha H_0^4 q_0^2 + 36H_0^4\alpha q_0)}{\alpha H_0} (t - t_0)^3 \\
 & + \frac{1}{2592} \frac{(t - t_0)^4}{\alpha H_0^2} \times \left(9H_0^4 + 162H_0^6\alpha - 12\mu_m^0 H_0^2 + 18H_0^4\alpha\mu_m^0 + 108H_0^4\alpha\mu_m^0 q_0 - 54H_0^6\alpha q_0^2 \right. \\
 & + 324H_0^6\alpha q_0 - 108\alpha H_0^6 q_0^3 - 6\mu_m^0 H_0^2 q_0 + 9\mu_m^0 \gamma_m H_0^2 + \mu_m^2 + 90\alpha\mu_m^0 H_0^4 q_0^2 - 12\alpha\mu_m^0{}^2 H_0^2 q_0 \\
 & \left. + 12\alpha\mu_m^2 H_0^2 + 108\mu_m^0 \gamma_m H_0^4\alpha - 108\mu_m^0 \gamma_m H_0^4\alpha q_0 \right) + O[(t - t_0)^5] . \quad (85)
 \end{aligned}$$

The above solution can be used to check observational constraints. As an alternative, we can also solve Eqn (84) numerically. A numerical solution for the Hubble parameter is developed in Fig. 1 where we put $a_0 = H_0 = 1$, $q_0 = -0.7$. We see in Fig. 1 that H is an oscillatory function, it reaches maxima and minima several times. It defines an oscillatory solution but it is not in the form of Type IV future singularity [48, 49, 50, 51, 52]. But it can be identified in the late-time as the Λ CDM era.

We can classify the future singularities as follows:

- Type I: (“Big Rip”): $t \rightarrow t_s$, $a \rightarrow \infty$, $\mu \rightarrow \infty$ and $|p| \rightarrow \infty$.
- Type II: (“sudden”): $t \rightarrow t_s$, $a \rightarrow a_s$, $\mu \rightarrow \mu_s$ and $|p| \rightarrow \infty$.
- Type III: $t \rightarrow t_s$, $a \rightarrow a_s$, $\mu \rightarrow \infty$ and $|p| \rightarrow \infty$
- Type IV: $t \rightarrow t_s$, $a \rightarrow a_s$, $\mu \rightarrow 0$ and $|p| \rightarrow 0$ and higher derivatives of H diverge. Here t_s , a_s and μ_s are constants with $a_s \neq 0$.

For our case, the factor given in Fig. 1, the Hubble parameter and first, second and third derivatives of H are plotted in Fig. 2. No higher derivatives of H diverges.

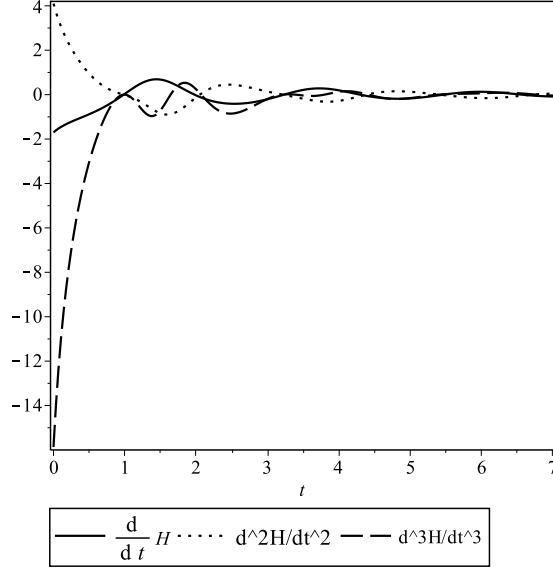


Fig. 2 Numerical solution for \dot{H} , \ddot{H} , \dddot{H} .

A phase portrait for Starobinsky model is plotted in Fig. 3. Here we solved the ODE with parameters $\Omega_m^0 \equiv \frac{\mu_m^0}{3H_0^2} = 0.3$, $\gamma_m = 1$. The phase portrait shows that the scale factor $a(t)$ is a monotonic increasing function of time. It is always increasing, and never decreasing.

For curiosity we are interested to know if the system has attractors or not. The late-time or asymptotic attractors are a class of solutions which have a generic form independent of the initial conditions. We examine our model for such types of solutions and solve the equations of motion for some initial conditions. The model is well established as an attractor in the following Fig. 4.

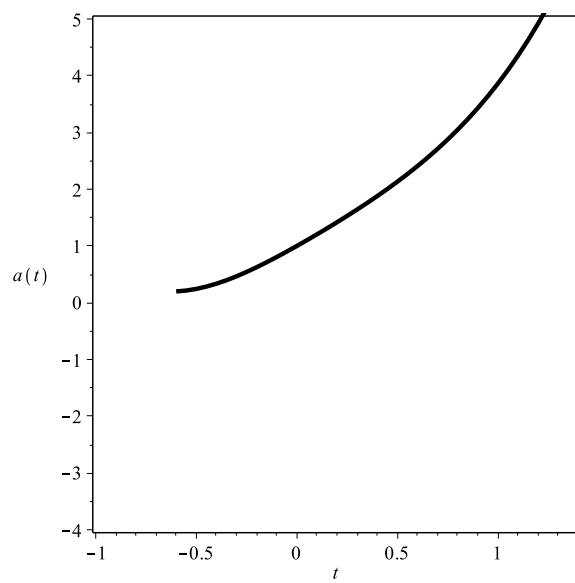


Fig. 3 Phase portrait for Starobinsky's model.

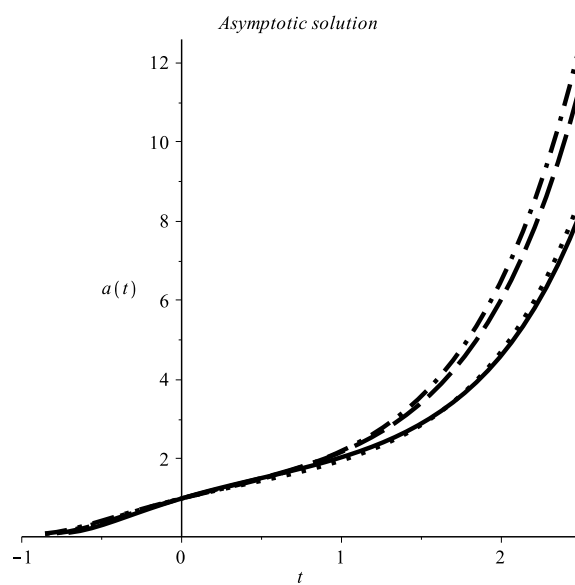


Fig. 4 Attractors for Starobinsky's model.

6 Discussions and Conclusion

In this work we looked at classes of shear-free anisotropic cosmological spacetimes in $f(R)$ gravity. Focusing on orthogonal models with irrotational and non-accelerated fluid flows without heat fluxes, we showed that the anisotropic stresses are related to the electric part of the Weyl tensor in such a way that they balance each other. This is considered necessary and sufficient condition for the shear to be vanishing forever if vanishing initially. This turned out to be a generalization of a previous result [16] for models whose underlying theory is GR. We also showed that within the class of orthogonal $f(R)$ models, small perturbations of shear are damped, i.e., that these models are stable if expanding, and that the electric part of the Weyl tensor and the anisotropic stress tensor decay with the expansion as well as the heat flux of the curvature fluid.

As an application, we considered a subclass of locally rotationally symmetric spacetimes with barotropic equations of state and studied the evolutionary dynamics of the Universe. In particular, we showed that the late-time behaviour of the dS universe in $f(R)$ gravity should satisfy Eqn (83). For the Starobinsky model of $f(R)$, we provided a power-series solution for $a(t)$ and we studied the behavior of the expansion parameter $H(t)$ by numerically integrating the Friedmann equation (84), where the initial conditions for H_0 , q_0 and j_0 are taken from observational data. The result is the oscillatory solution presented in Fig. 1 and describes the late-time universe in the Λ CDM era. The first three derivatives of H have also been calculated as shown in Fig. 2; none of these derivatives diverges. A phase-portrait analysis for this model with $\Omega_m^0 = 0.3$, $\gamma_m = 1$, given in Fig. 3, shows that the scale factor is a monotonically increasing function of time. Finally, we examined our model for late-time or asymptotic attractors, with well established solutions depicted in Fig. 4.

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